Math. Ann. 292, 267–280 (1992)

Mathematische Annalen

The Schwarzian derivative and conformally natural quasiconformal extensions from one to two to three dimensions*

Martin Chuaqui[1] and Brad Osgood[2]

[1] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
[2] Department of Mathematics, Stanford University, Stanford, CA 94305, USA

Received February 20, 1991; in revised form September 17, 1991

1 Introduction

The growth of the Schwarzian derivative of an analytic function is related to both injectivity and quasiconformal extension of the function. This connection was first discovered by Ahlfors and Weill [AW], who generalized an injectivity criterion of Nehari [N]. They proved that if f is analytic and locally injective in the unit disk and if for some $t<1$ the Schwarzian derivative Sf satisfies

$$(1.1)\qquad (1-|z|^2)^2|Sf(z)| \leqq 2t\,,$$

then f is injective in the disk and has a K-quasiconformal extension to $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where K depends only on t. The Schwarzian is defined by

$$(1.2)\qquad Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

If $2t$ is replaced by 2 in (1.1) then one obtains Nehari's original injectivity criterion. Nehari's result started what is by now a considerable amount of work in this area, see, e.g. [L].

Each of the present authors has been interested in theorems of this type, expecially in generalizations to higher dimensions, [OS1, 2, C1, 2]. What about *one* dimension? For a smooth real valued function on an interval, the Schwarzian can again be defined by (1.2), but there is no question of proving an injectivity criterion since one must already assume that $f' \neq 0$ to define Sf. Remarkably, what does persist is the phenomenon of quasiconformal extension. We shall prove the following.

* To Professor F. W. Gehring on his 65$^{\text{th}}$ birthday

Theorem 1. *If f is a C^3 function on $(-1,1)$, $f'\neq 0$, and if for some $0\leqq t<1$ one has*

$$\frac{-4t}{1-t}\leqq(1-x^2)^2 Sf(x)\leqq\frac{4t}{1+t} \tag{1.3}$$

then f has a conformally natural $K(t)=\frac{1+t}{(1-t)^3}$-quasiconformal extension to the plane and to space, preserving the upper half-plane and upper half-space, respectively.

By *conformally natural* we mean in the sense of Douady and Earle [DE]. Let $E_2(f)$ and $E_3(f)$ denote the extensions of f to two and three dimensions. If A is a Möbius transformation of $\mathbb{C}$ preserving the real axis, or of $\mathbb{H}^3$ preserving the vertical plane through the real axis, then the extensions have the property that

$$A\circ E_j(f)=E_j(A\circ f). \tag{1.4}$$

We do not know how our extensions compare with those of Douady and Earle, who consider extensions of homeomorphisms of the circle. The extension $E_2(f)$ agrees with the Ahlfors-Weill extension on the real axis.

The restriction of $E_2(f)$ to the real axis gives an extension $E_1(f)$ of f to $\bar{\mathbb{R}}$ which is the boundary value function of a quasiconformal mapping of the upper half-plane. We would thus say that f has a quasisymmetric extension to the real line (see [L]) were it not for the fact that $E_1(f)$ might not fix ∞, as is assumed in the definition of quasisymmetry. However if $A(x)=\frac{ax+b}{cx+d}$, $a, b, c, d\in\mathbb{R}$ is a Möbius transformation of the line then the invariance property of the Schwarzian, $S(A\circ f)=S(f)$, and the conformal naturality of the extensions imply that at least some Möbius transformation of f has a quasisymmetric extension. In fact, one possibility is $A\circ f$ where $(A\circ f)''(0)=0$, as we shall see in Sect. 4.

Of course the choice of the interval $(-1,1)$ is unimportant, and though we have found it more convenient to work on the real line, one can formulate the theorem and its proof for functions defined on an arc of the circle. The Schwarzian can be defined as before and if f is a smooth function on, say, the upper semicircle of $|z|=1$, then the equivalent formulation of (1.3) is

$$-\frac{4t}{1-t}\leqq 2(1-\cos 2\theta)Sf(\theta)\leqq\frac{4t}{1+t}, \tag{1.5}$$

calling the coordinate θ. One then has a quasiconformal extension to the disk and the ball.

Our work is an application of the beautiful ideas and methods of Epstein [E1, 2]. Though an ingenious use of reflections in complete surfaces in hyperbolic 3-space, he showed in a new and elegant way how to relate the Schwarzian to quasiconformal extensions. We follow his lead, but we generate the surfaces we use by rotating a curve in the hyperbolic plane, giving a surface of revolution in the upper half-space. The effect is to drop down one dimension and the trick, not so immediate, is to make Epstein's formulas applicable when, to begin with, one has estimates only along the generating curve.

Finally, we point out that the image of the unit disk under the extension $E_2(f)$ and the image of the unit ball under the extension $E_3(f)$ give nice examples of quasidisks and domains quasiconformally equivalent to a ball, respectively. The latter are sometimes harder to come by, and since the mappings we build of f really are quite explicit, we hope this may be of further interest.

2 Support functions and quasiconformal reflections

We will use the upper half-space $\mathbb{H}^3$ as a model for hyperbolic 3-space. There are two main facts at the center of Epstein's work. First, a complete, imbedded surface Σ in $\mathbb{H}^3$ whose principal curvatures k_1, k_2 in the hyperbolic metric are <1 in absolute value has a Jordan curve $\partial\Sigma \subset \bar{\mathbb{C}}$ as its boundary at infinity. (We may also assume that $\partial\Sigma$ is bounded). Second, the surface determines an orientation reversing quasiconformal reflection $\Lambda: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ across $\partial\Sigma$. This reflection is easy to describe geometrically. Let Ω be the bounded component of $\mathbb{C} \backslash \partial\Sigma$, so that Σ lies above Ω. The unit normal vector field to Σ, say the exterior normal, determines at each point $P \in \Sigma$ a unique geodesic in $\mathbb{H}^3$, normal to Σ at P, with one endpoint in Ω and one in $\bar{\mathbb{C}} \backslash \Omega$. The map $\Lambda: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ pairs these endpoints, say $\zeta \in \Omega$ to $\Lambda(\zeta) \in \bar{\mathbb{C}} \backslash \Omega$.

Proving that this works depends upon recovering Σ as the envelope of a family of horospheres. There is a particularly useful way of parametrizing the horospheres that brings to light a relation between the principal curvatures and the Beltrami coefficient of the reflection Λ. A horosphere H in $\mathbb{H}^3$ is determined by the point where it is tangent to $\partial\mathbb{H}^3$, which we refer to as its base, and by its *horospheric radius* ϱ. The latter is defined to be

$$\varrho = \pm \inf_{P \in H} d(P, (0,0,1)) \tag{2.1}$$

where $d(.,.)$ denotes the hyperbolic distance in $\mathbb{H}^3$ and ϱ is taken to be positive if $(0,0,1)$ is outside of H and negative if it is inside, in the obvious sense. Now, quite generally, one can define a surface in $\mathbb{H}^3$ over a domain $\Omega \subset \mathbb{C}$ as the envelope, in the usual sense of differential geometry, of a family of horospheres $H(\zeta, \varrho(\zeta))$ where $\zeta \in \Omega$ is the base and $\varrho(\zeta)$ is the horospheric radius. Under appropriate hypotheses on ϱ the envelope of such a family of horospheres will be a complete, imbedded surface Σ in $\mathbb{H}^3$ with bounds on the principal curvatures as described in the last paragraph. The function ϱ is then called the *support function* of the surface. Conversely, such a surface over a domain Ω is the envelope of the family of its tangent horospheres and so determines a support function ϱ on Ω.

The bounds on the principal curvatures of Σ and the quasiconformality of the reflection Λ across $\partial\Sigma = \partial\Omega$ occur simultaneously, so to speak. For if

$$\psi(\zeta) = \varrho(\zeta) - \log(1 + |\zeta|^2), \quad \zeta \in \Omega, \tag{2.2}$$

then the Beltrami coefficient of the reflection Λ is computed to be

$$\mu = \frac{\psi_{\zeta\zeta} - (\psi_\zeta)^2}{\psi_{\zeta\bar{\zeta}}}, \tag{2.3}$$

and the bound

$$|\mu(\zeta)| < 1 \tag{2.4}$$

turns out to be equivalent to the bound

$$\max\{|k_1(P)|, |k_2(P)|\} < 1 \tag{2.5}$$

on the principal curvatures.

3 Surfaces of revolution in $\mathbb{H}^3$

We use the upper half-plane $\mathbb{H}^2$ as a model for the hyperbolic plane. We regard $\mathbb{H}^2$ as isometrically imbedded in $\mathbb{H}^3$ as a vertical half-plane; say $\partial\mathbb{H}^2$ is the real axis in $\mathbb{C}=\partial\mathbb{H}^3$. Let Γ be a complete, immersed smooth curve in $\mathbb{H}^2$ with hyperbolic geodesic curvature k satisfying $|k|<1$. Then Γ is actually imbedded [E2, p. 21] with (finite) asymptotic endpoints on $\partial\mathbb{H}^2$. Thinking of $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$ as above, let Σ be the surface in $\mathbb{H}^3$ obtained by rotating Γ about the real axis by 90° in both directions. Then Σ is a complete imbedded surface in $\mathbb{H}^3$. The *longitudes* of Σ are copies of the curve Γ, though they are not isometric to Γ in the hyperbolic metric. The *latitudes* are the orthogonal semicircles, which are geodesics in $\mathbb{H}^3$.

Σ has two principal directions and two corresponding principal curvatures k_1, k_2 at each point. Recall that X_i, $i=1,2$ is a principal direction and k_i the corresponding principal curvature if $\nabla_{X_i}N=k_iX_i$. Here ∇ is the Riemannian connection in $\mathbb{H}^3$ and N is the exterior unit normal to Σ. Because the euclidean and hyperbolic metrics are conformal, the lines of curvature of Σ are the same in both metrics, namely the latitudes and longitudes. This is straightforward to check using the formula for the change in the connection under a conformal change in the metric, see e.g. [OS1, Sect. 2]. Since the latitudes are geodesics in $\mathbb{H}^3$, the principal curvature along these curves, say k_1, is identically zero. The principal curvature along the (vertical) generator curve Γ is its hyperbolic geodesic curvature in $\mathbb{H}^2$. We shall need to know that the principal curvature k_2 along *any* longitude is no greater in absolute value than its value along Γ. We can then conclude that for any $P\in\Sigma$,

$$\max\{|k_1(P)|,\ |k_2(P)|\}=|k_2(P)|<1,$$

making the results of the previous section applicable.

Lemma 1. *Let Γ' be a longitude, P' a point on Γ' and P the corresponding point on Γ on the latitude through P'. Then $|k_2(P')|\leqq|k(P)|$.*

Proof. Let $\delta(.)$ denote the euclidean distance from a point in $\mathbb{H}^3$ to $\partial\mathbb{H}^3$; the hyperbolic metric in $\mathbb{H}^3$ is then $\delta^{-1}\cdot$(euc). Let k_e denote the euclidean curvature of Γ, and hence also of Γ'. The principal curvature $k_2(P')$ at P' in the direction of Γ' is related to the euclidean curvature $k_e(P')$ by

$$k_2(P')=\delta(P')\left(k_e(P')+\frac{\partial}{\partial n}\left(\log\frac{1}{\delta}\right)(P')\right),$$

see e.g. [OS1, Sect. 3]. Here $\frac{\partial}{\partial n}$ denotes the (euclidean) exterior normal derivative on Σ.

When $\Gamma'=\Gamma$ the principal curvature $k_2(P')=k(P)$. Next, the euclidean curvature $k_e(P')$ is constant along the latitude from P to P' and it is easy to check that $\frac{\partial}{\partial n}\left(\log\frac{1}{\delta}\right)(P')$ is also constant along the latitude. Since the vertical distance δ decreases from P to P' we have $|k_2(P')\leqq|k(P)|$ as desired. In fact, the calculation actually gives the rather nice formula

$$k_2(P')=\frac{\delta(P')}{\delta(P)}\,k(P).$$

The surface Σ lies over a bounded, symmetric, Jordan domain $\Omega \subset \mathbb{C}$, $\partial\Sigma = \partial\Omega$, whose boundary consists of a copy of the curve Γ and its reflection in the real axis. As described in the previous section, we recover Σ as the envelope of the family of its tangent horospheres $H(\zeta, \varrho(\zeta))$, $\zeta \in \Omega$, where ϱ is the support function of Σ. Write $\zeta = u + iv$. Then because of the symmetry $\varrho(\zeta) = \varrho(u, v)$ is even in v. For each $\zeta \in \Omega$ we let $T(\zeta)$ (for tangent) denote the point on Σ when $H(\zeta, \varrho(\zeta))$ is tangent to Σ. By using a Möbius transformation of $\mathbb{H}^3$ we can assume without loss of generality that $0 \in \Omega$ and that $T(0) = (0, 0, 1)$.

The general relation given by Epstein between the Beltrami coefficient of the reflection Λ and the principal curvatures of the surface reduces here, in the case of a surface of revolution, to

$$\frac{1 + |\mu(\zeta)|}{1 - |\mu(\zeta)|} = \frac{1 + |k_2(T(\zeta))|}{1 - |k_2(T(\zeta))|}$$

i.e., to

$$|\mu(\zeta)| = |k_2(T(\zeta))|. \tag{3.1}$$

Hence by Lemma 1, $|\mu|$ will be no greater on Ω than its value on the interval $\Omega \cap \mathbb{R}$, under the generating curve Γ. Our goal is to compute μ along this interval in terms of $\varrho(u, 0)$ and its u-derivatives alone.

Recall from the previous section that if

$$\psi(\zeta) = \varrho(\zeta) - \log(1 + |\zeta|^2) \tag{3.2}$$

then

$$\mu = \frac{\psi_{\zeta\zeta} - \psi_\zeta^2}{\psi_{\zeta\bar{\zeta}}}. \tag{3.3}$$

Because ϱ is even in v the only difficult term to compute is $\psi_{vv}(u, 0)$. This is an interesting calculation involving a mix of the hyperbolic and euclidean geometry of horospheres. We present it as a separate lemma.

From the symmetry of Σ it is clear that the reflection Λ maps the real axis $v = 0$ to itself and preserves the half-planes $v > 0$ and $v < 0$. Let $R(u) = \frac{1}{2}(\Lambda(u) - u)$ for $u \in \Omega \cap \mathbb{R}$, and for $\zeta \in \Omega$ let $r(\zeta)$ denote the euclidean radius of the horosphere $H(\zeta, \varrho(\zeta))$. One finds that

$$\varrho(\zeta) = \log\left(\frac{1 + |\zeta|^2}{2r(\zeta)}\right). \tag{3.4}$$

We shall prove

Lemma 2. *With the notation as above,*

$$\psi_{vv}(u, 0) = \frac{1}{2}\left(\frac{1}{r(u)^2} - \frac{1}{R(u)^2}\right), \qquad u \in \Omega \cap \mathbb{R}.$$

Proof. Recall we have assumed that Σ passes through $(0, 0, 1)$ and that this is the point of tangency between Σ and $H(0, \varrho(0))$, [so $\varrho(0) = 0$]. The vertical semicircle, say $v^2 + \xi^2 = 1$, $\xi > 0$, is then a latitude of Σ and $\varrho(0, v) = \log(1 + v^2) - \log(1 - v^2)$. Hence $\psi(0, v) = -\log(1 + v^2)$ and $\psi_{vv}(u, 0) = 2$. Since $r(0) = \frac{1}{2}$ and $R(0) = \infty$ the formula checks in this case. The origin is the only point with $R = \infty$ since Λ is 1 : 1.

Let A denote the origin, take $B=(0,\sigma,0)$, $|\sigma|<1$, to be a variable point on the v-axis with $C=(0,1,0)\in\partial\Omega$. Let $D=T(B)$, the point of tangency between Σ and $H(B,\varrho(B))$. Then

$$D=\left(0,\frac{2\sigma}{1+\sigma^2},\frac{1-\sigma^2}{1+\sigma^2}\right). \tag{3.5}$$

Finally, let $E=(0,0,|\sigma|)$, so that the hyperbolic distance from $(0,0,1)$ to E is $-\log|\sigma|$.

Now fix $U_0=(u_0,0,0)$ and choose a Möbius transformation of $\mathbb{H}^3$ mapping $H(A,\varrho(A))$ to $H(u_0,\varrho(u_0))$ and preserving $\mathbb{H}^2\subset\mathbb{H}^3$. Let B', C', D', and E' be the points corresponding to B, C, D, E under this map. The image of the segment AC is a circular arc (in $\mathbb{C}$) from U_0 through B' to C', orthogonal to the real axis and parametrized by σ. We will be able to compute the second derivative of ψ along this arc at U_0. This derivative contains the term $\psi_{vv}(u_0,0)$ giving is what we want. Thus the essence of the proof is to parametrize at the origin and perturb by a Möbius transformation.

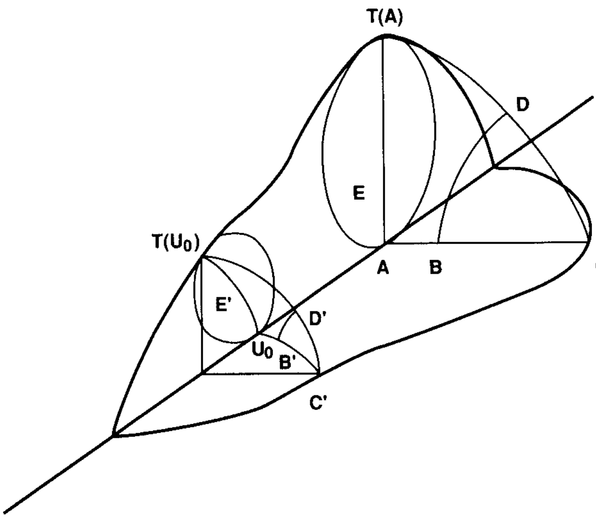

Fig. 1

We need the coordinates for $T(U_0)$, B' and D'. For $T(U_0)$, the point of tangency of Σ and $H(U_0,\varrho(U_0))$, we find that $T(U_0)=(t_0,0,s_0)$ where,

$$t_0=u_0+\frac{2R_0r_0^2}{R_0^2+r_0^2},\qquad s_0=\frac{2R_0^2r_0}{R_0^2+r_0^2}, \tag{3.6}$$

writing $r_0=r(U_0)$, $R_0=R(U_0)$. Since the hyperbolic distance between $T(U_0)$ and E' is also $-\log|\sigma|$ this allows us to find $E'=(1,0,b)$ with

$$a=u_0+\frac{2R_0r_0^2\sigma^2}{R_0^2+r_0^2\sigma^2},\qquad b=\frac{2R_0^2r_0|\sigma|}{R_0^2+r_0^2\sigma^2}. \tag{3.7}$$

The geodesic in $\mathbb{H}^3$ from B to E goes into a geodesic from B' to E' meeting $\mathbb{H}^2$ at E' at a right angle. Hence $B'=(a, \pm b, 0)$ with a and b as in (3.7). Finally we find that $D'=T(B')$ has coordinates

$$D'=\left(t_0, \frac{2\sigma s_0}{1+\sigma^2}, \frac{1-\sigma^2}{1+\sigma^2}\, s_0\right)=(t_0, \alpha, \beta) \tag{3.8}$$

where t_0 and s_0 are as in (3.6).

The horosphere $H(B', \varrho(B'))$ is tangent to the surface Σ at D'. From (3.4) we relate the euclidean and horospheric radius $r(B')$ and $\varrho(B')$ by

$$\varrho(B')=\log\left(\frac{1+|B'|^2}{2r(B')}\right) \tag{3.9}$$

and hence for ψ, as defined by (3.2), we get

$$\psi(B')=-\log r(B')-\log 2. \tag{3.10}$$

B' moves along the circular arc U_0C' parametrized by σ, so we consider r and ψ to be functions of σ with $\sigma=0$, corresponding to $B'=U_0=(u_0, 0, 0)$. Since, by symmetry, $r_\sigma(0)=0$ we have from (3.10) that

$$\psi_{\sigma\sigma}(0)=\frac{1}{r(0)}\, r_{\sigma\sigma}(0)=\frac{1}{r_0}\, r_{\sigma\sigma}(0). \tag{3.11}$$

Next, if $D''=\left(t_0, \frac{2\sigma s_0}{1+\sigma^2}, 0\right)$ is the projection of D' onto the plane and h is the euclidean distance between B' and D'' then

$$h^2=(\alpha-b)^2+(t-a)^2$$

and

$$h^2=(\beta-r)^2+r^2$$

whence

$$2\beta r=(\alpha-b)^2+(t_0-a)^2+\beta^2.$$

This last expression enables us to find $r_{\sigma\sigma}(0)$. After some computation the final result simplifies to

$$r_{\sigma\sigma}(0)=-4\,\frac{r(0)^2}{s_0}. \tag{3.12}$$

On the other hand the chain rule gives that at $\sigma=0$

$$\psi_{\sigma\sigma}(0)=4r_0^2\left(\psi_{vv}(u_0, 0)+\frac{1}{R_0}\,\psi_u(u_0, 0)\right). \tag{3.13}$$

It remains to compute $\psi_u(u_0, 0)$. Using (3.9) and a formula of Epstein for the reflection, [E1, p. 123] we can write

$$R(u)=\frac{1}{2}\,(\Lambda(u)-u)=\frac{1+u^2}{(1+u^2)\varrho_u-2u}. \tag{3.14}$$

Then $\psi(u,0)=\varrho(u,0)-\log(1+u^2)$ gives $\psi_u(u_0,0)=\dfrac{1}{R_0}$ and with (3.14) this leads to

$$\psi_{vv}(u_0,0)=\frac{1}{4r_0^2}\,\psi_{\sigma\sigma}(0)-\frac{1}{R_0^2}.$$

This together with (3.11) and (3.12) complete the proof of Lemma 2.

With this accomplished we return to the Beltrami coefficient μ of the reflection Λ as given in (3.3). Lemma 2 and Eq. (3.14) used in the proof lead easily to the expression

$$\mu(u,0)=\frac{(1+u^2)^2(\varrho_{uu}-\frac{1}{2}\varrho_u^2)+2u(1+u^2)\varrho_u-2-2e^{2\varrho(u)}}{(1+u^2)^2(\varrho_{uu}-\frac{1}{2}\varrho_u^2)+2u(1+u^2)\varrho_u-2+2e^{2\varrho(u)}} \tag{3.15}$$

on the interval $\Omega\cap\mathbb{R}$. As we know, $|\mu|<1$ on Ω if and only if $|\mu(u,0)|<1$ on $\Omega\cap\mathbb{R}$, that is, if and only if

$$(1+u^2)^2(\varrho_{uu}-\tfrac{1}{2}\varrho_u^2)+2u(1+u^2)\varrho_u>2.$$

The condition on ϱ leading to $|\mu|\leqq t<1$ will come up in the proof of Theorem 1 in the next section. Let us indicate briefly how the work in this section is applied there. A function f defined on the interval $(-1,1)$ and satisfying the hypothesis of Theorem 1 will be used to define a function ϱ on the *image* of f, which we can assume is also $(-1,1)$. This ϱ will be the support function for a *curve* Γ in $\mathbb{H}^2\subset\mathbb{H}^3$. Geometrically, such a support function is defined just as for surfaces, but this time Γ is realized as the envelope of a family of *horocycles* $H(x,\varrho(x))\cap\mathbb{H}^2$ in $\mathbb{H}^2$.

The formulas we have used, and have derived, for geometric quantities, μ, etc., are all local and hence Eq. (3.15) above actually gives a formula for the hyperbolic geodesic curvature of Γ in $\mathbb{H}^2$ in terms of its support function. The hypothesis of Theorem 1 will imply that Γ is complete and that its curvature is $\leqq t<1$. We then rotate Γ to produce a surface of revolution Σ in $\mathbb{H}^3$ of the type we have been discussing over a domain $\Omega\subset\mathbb{C}$. The symmetric extension of ϱ to Ω is the support function for Σ. We also then get a quasiconformal reflection Λ across $\partial\Omega$ which will be used to extend f.

4 Proof of Theorem 1

Suppose f is a C^3 function on $(-1,1)$ satisfying

$$-\frac{4t}{1-t}\leqq(1-x^2)^2Sf(x)\leqq\frac{4t}{1+t},\qquad x\in(-1,1) \tag{4.1}$$

for some $0\leqq t<1$. Since $f'\neq0$ we may assume that $f'>0$, and hence that f is increasing on $(-1,1)$. Using the invariance property of the Schwarzian, $S(A\circ f)=S(f)$ for a Möbius transformation A, we can further normalize f to satisfy $f(0)=0$, $f'(0)=1$, $f''(0)=0$. It follows from the work in [CO] that with this normalization f is subject to the sharp bounds

$$|f(x)|\leqq\frac{1}{\sqrt{1-s}}\,\frac{(1+x)^{\sqrt{1-s}}-(1-x)^{\sqrt{1-s}}}{(1+x)^{\sqrt{1-s}}+(1-x)^{\sqrt{1-s}}},$$

$$|f'(x)|\leqq\frac{4(1-x^2)^{\sqrt{1-s}-1}}{((1+x)^{\sqrt{1-s}}+(1-x)^{\sqrt{1-s}})^2},$$

where $2s=4t/(1+t)$. [Actually, only the upper bound in (4.1) is needed for this.] One important consequence is that f is bounded. Without affecting either this or the order of f' we may then further shift and scale to get that $f(-1)=-1$ and $f(1)=1$; this is just for convenience.

Let $h=f^{-1}$, $u=f(x)$, $x=h(u)$, and define ϱ on $(-1,1)$ through the pullback by h of the "Poincaré metric" of $(-1,1)$. That is, we define ϱ by the equation

$$\frac{e^{2\varrho(u)}}{(1+u^2)^2}\,du^2=h^*\left(\frac{dx^2}{(1-x^2)^2}\right),$$

or

$$\varrho(u)=\log(1+u^2)+\log h'(u)-\log(1-h(u)^2)\,. \tag{4.2}$$

It then follows from the estimates above that ϱ tends to $+\infty$ as $u\to\pm 1$.

Computing $\mu(u,0)$ according to (3.15) and using

$$S(h)(u)=-Sf(x)h'(u)^2\,,$$

shows directly that the condition (4.1) and the bound $|\mu(u,0)|\leqq t<1$ are equivalent statements. Hence if (4.1) holds the envelope curve $\Gamma\subset\mathbb{H}^2\subset\mathbb{H}^3$ of the family of horocycles $H(u,\varrho(u))\cap\mathbb{H}^2$, $u\in(-1,1)$ has hyperbolic geodesic curvature bounded in absolute value by $t<1$. Since $\varrho(u)\to\infty$ at the endpoints, Γ is complete and imbedded and the work of the previous section is applicable. Rotating Γ then gives a surface $\Sigma\subset\mathbb{H}^3$ lying over a bounded, symmetric domain $\Omega\subset\mathbb{C}$, with $\partial\Sigma=\partial\Omega$, and a quasiconformal reflection $\Lambda:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ across $\partial\Omega$, which is even in v, whose Beltrami coefficient μ on Ω is bounded by t. The support function for Σ extends ϱ to Ω as an even function of v, and we continue to denote the extension by ϱ.

We first extend f to a quasiconformal mapping $\tilde{f}$ mapping the unit disk onto Ω. Let Ω_0 denote the unit disk, and let Σ_0 denote the hemisphere in $\mathbb{H}^3$ over Ω_0 with $\partial\Sigma_0=\partial\Omega_0$. Then Σ_0 can be recovered as the envelope of its tangent horospheres $H_0(z,\varrho_0(z))$, $z\in\Omega_0$. Here, the support function ϱ_0 is simply

$$\varrho_0(z)=\log\frac{1+|z|^2}{1-|z|^2}\,. \tag{4.3}$$

For $z\in\Omega_0$, let $T_0(z)$ be the point on Σ_0 where Σ_0 is tangent to $H_0(z,\varrho_0(z))$. The mapping $z\mapsto T_0(z)$ is conformal. (Geometrically, this is essentially because Σ_0 is totally geodesic as a submanifold of hyperbolic space.) Longitudes on Σ_0 are semicircles rotated about the x-axis and latitudes are the orthogonal semicircles. Let Γ_0 denote the vertical longitude.

For the surface Σ lying over Ω, the map $z\mapsto T(z)$ of Ω onto Σ is $(1-t)^{-1}$-quasiconformal. By this we mean that the ratio of the largest eigenvalue to the smallest eigenvalue of the derivative map is bounded above by $(1-t)^{-1}$ on Ω. Again, this is a consequence of the calculations in [E1, p. 119].

We define a map $\tilde{f}$ of $\bar{\Omega}_0$ onto $\bar{\Omega}$ as follows. Starting with $z\in\bar{\Omega}_0$, the point $T_0(z)$ is on a longitude at some angle $\theta\in[-\pi/2,\pi/2]$ to the vertical. Rotate the longitude back to the vertical longitude Γ_0 giving a point $T_0(x)$, for some $x\in[-1,1]$. The image $f(x)\in[-1,1]$ then determines a point $T(f(x))$ on $\bar{\Gamma}$, the vertical longitude of Σ. Rotate this longitude back through the same angle θ giving a point $T(\zeta)$ on $\bar{\Sigma}$ for some $\zeta\in\bar{\Omega}$. The map $\tilde{f}$ is the correspondence $z\mapsto\zeta$. $\tilde{f}$ is smooth except perhaps at the two points where $\partial\Omega$ meets the real axis. From the remarks above on the maps T_0 and T, to show that $\tilde{f}$ is quasiconformal we must show that the intermediate map $\Phi:\bar{\Sigma}_0\to\bar{\Sigma}$, $T_0(z)\mapsto T(\zeta)$, is quasiconformal.

Write $P=T_0(z)$, $Q=T(\zeta)$, and $Q=\Phi(P)$. For a point $P\in\Sigma_0$ the eigenvectors of the derivative $D\Phi(P)$ are along the longitude and latitude through P. The eigenvalue corresponding to the latitudinal direction is simply $l(Q)/l_0(P)$ where $l(Q)$ and $l_0(P)$ are the euclidean radii of the latitudes on Σ and Σ_0 through Q and P respectively. Furthermore the eigenvalue in the longitudinal direction will be constant along the entire latitude through P, and therefore it suffices to compute it when $P\in\Gamma_0$.

Let g_0 be the euclidean metric and let g be the hyperbolic metric on $\mathbb{H}^2\subset\mathbb{H}^3$. For $P\in\Gamma_0$, $Q=\Phi(P)\in\Gamma$ write $P=T_0(x)$, $x\in(-1,1)$ and $Q=T(u)$, $u=f(x)\in(-1,1)$. Along $(-1,1)$ let $g=e^{2\varphi}|dx|^2$ with $\varphi(x)=-\log(1-x^2)$ and $g_2=e^{2\psi}|du|^2$ with $\psi(u)=\log h'(u)-\log(1-h(u)^2)$, $h=f^{-1}$. Denoting norms of the differentials in the various metrics by $|\cdot|_{g_0}$, etc., we first have [E2, p. 23]

$$|dx|_{g_1}=2|dP|_g,\qquad |du|_{g_2}=2(1-k(Q))\,|dQ|_g, \tag{4.4}$$

Here, recall that k is the hyperbolic geodesic curvature of Γ. Also

$$|du|_{g_0}=f'(x)\,|dx|_{g_0}. \tag{4.5}$$

Next, since $P\in\Gamma_0$, $Q\in\Gamma$, the radii $l_0(P)$ and $l(Q)$ are the euclidean distances from P and Q to $\mathbb{C}$, respectively. Therefore, for the hyperbolic metric

$$|dP|_g=\frac{1}{l_0(P)}\,|dP|_{g_0},\qquad |dQ|_g=\frac{1}{l(Q)}\,|dQ|_{g_0}. \tag{4.6}$$

Combining Eq. (4.4) through (4.6) now gives

$$\frac{|dQ|_{g_0}}{|dP|_{g_0}}=(1-k(Q))^{-1}\,\frac{l(Q)}{l_0(P)}. \tag{4.7}$$

This is the eigenvalue of $D\Phi(P)$ at $P\in\Gamma_0$ in the direction of Γ_0 and hence along the entire latitude through P. Since $|k(Q)|\leqq t<1$, taking the ratio of the eigenvalues in the two directions we conclude that the dilatation of Φ is at most $(1-t)^{-1}$, (even on $\bar{\Sigma}_0$).

Recall that the map $\tilde{f}$ which extends f to be a mapping of $\bar{\Omega}_0$ onto $\bar{\Omega}$ is defined by $\tilde{f}=T^{-1}\circ\Phi\circ T_0$. It is therefore $(1-t)^{-2}$-quasiconformal. To extend f to be a quasiconformal mapping of $\mathbb{C}$ we now define

$$E_2(f)(z)=\begin{cases}\tilde{f}(z), & z\in\bar{\Omega}_0,\\ \Lambda\left(\tilde{f}\left(\dfrac{1}{\bar{z}}\right)\right), & z\notin\bar{\Omega}_0.\end{cases}$$

Since Λ is a $\dfrac{1+t}{1-t}$-quasiconformal reflection, $E_2(f)$ is a $\dfrac{1+t}{(1-t)^3}$-quasiconformal mapping of $\mathbb{C}$. It maps Ω_0 onto Ω showing that Ω is a quasidisk.

The symmetry implies that $E_2(f)$ preserves the upper and lower half-planes and hence $E_1(f)=E_2(f)|\mathbb{R}$ maps the real axis onto itself. It is a quasisymmetric extension of f, as discussed in Sect. 1.

Next, we show how to extend $E_2(f)$ to a quasiconformal mapping first of $\mathbb{H}^3$ and then by reflection to all of space. Recall that the quasiconformal reflection Λ across $\partial\Omega=\partial\Sigma$ pairs the endpoints $\zeta\in\Omega$ and $\Lambda(\zeta)\in\bar{\mathbb{C}}\backslash\Omega$ of the geodesic in $\mathbb{H}^3$ which is normal to Σ at the point $T(\zeta)$. We consider the geodesic flow G^τ, $\tau\in(-\infty,\infty)$, $G^0=\mathrm{id}$, from Σ. That is, starting at $P=T(\zeta)$ on Σ, $G^\tau(P)$ is the point on the geodesic through ζ, P, and $\Lambda(\zeta)$ that is a hyperbolic distance $|\tau|$ from P; positive τ flows toward $\zeta\in\Omega$ and negative τ flows toward $\Lambda(\zeta)\in\overline{\mathbb{C}}\backslash\Omega$. This gives a family of parallel surfaces $\Sigma^\tau=G^\tau(\Sigma)$ foliating $\mathbb{H}^3$. This flow was studied by Epstein [E2, p. 18] in a more general setting, and for the surface of revolution Σ one consequence of his work is that the map $G^\tau:\Sigma\to\Sigma^\tau$ is quasiconformal with dilatation

$$K(G^\tau)=\max\left\{\left|\frac{e^{2\tau}+1}{e^{2\tau}(1-k_2)+(1+k_2)}\right|, \left|\frac{e^{2\tau}(1-k_2)+(1+k_2)}{e^{2\tau}+1}\right|\right\}.$$

As before, k_2 is the principal curvature of Σ in the longitudinal direction ($k_1=0$) and since $|k_2|\leqq t<1$ we easily find that

$$K(G^\tau)\leqq\frac{1+t}{(1-t)^2}$$

for all τ. Moreover, for $P\in\Sigma$ it makes sense to form the limits $\zeta=\lim_{\tau\to\infty}G^\tau(P)$, $\zeta'=\lim_{\tau\to-\infty}G^\tau(P)$, the limits taken along the geodesic, and $\zeta'=\Lambda(\zeta)$.

We also form the geodesic flow G_0^τ for the hemisphere Σ_0 lying over the unit disk Ω_0. In this case the maps $G_0^\tau:\Sigma_0\to\Sigma_0^\tau$ are all conformal. Again the parallel surfaces Σ_0^τ folliate $\mathbb{H}^3$.

We extend $E_2(f):\overline{\mathbb{C}}\to\overline{\mathbb{C}}$, hence f, to $\mathbb{H}^3$ as follows. If $P\in\mathbb{H}^3$ then $P\in\Sigma_0^\tau$ for some τ, hence $G_0^{-\tau}(P)\in\Sigma_0$. Using the map $\Phi:\Sigma_0\to\Sigma$ and $G^\tau:\Sigma\to\Sigma^\tau$ we simply set

$$E_3(f)(P)=\begin{cases}(G^\tau\circ\Phi\circ G_0^{-\tau})(P), & P\in\Sigma_0^\tau\\ E_2(f)(P), & P\in\overline{\mathbb{C}}.\end{cases}$$

As G_0^τ is conformal, Φ is $(1-t)^{-1}$-quasiconformal, G^τ is $\dfrac{1+t}{(1-t)^2}$-quasiconformal and preserves hyperbolic lengths in the direction of the flow, we conclude that $E_2(f)$ is a $\dfrac{1+t}{(1-t)^3}$-quasiconformal mapping of $\mathbb{H}^3$ extending $E_2(f)$. By reflection $E_3(f)$ extends to a quasiconformal mapping of $\overline{\mathbb{R}}^3$ with the same dilatation. It maps the ball onto the solid bounded by Σ and its reflection in $\overline{\mathbb{C}}$.

Before continuing, we note again that the normalization on f and the consequent bounds allow us to construct directly the complete curve Γ, the complete surface Σ, and the quasiconformal reflection Λ. Instead of normalizing in this way one could also consider dilates $f_s(x)=f(sx)$, $s<1$, of f and apply a normal families argument to obtain a quasiconformal map which is used to extend f. This is what Epstein does in the case of functions analytic in the disk. However, as be points out, in the general situation he considers, there is no guarantee a priori that the limiting surface will be complete. This means that the limiting quasiconformal map is not necessarily induced as a reflection in a complete surface. We prefer the approach here because one gets Σ without passing to a limit. It also shows that in this case the surfaces associated to the dilates f_s actually do converge to a complete surface.

Next we need to show that the extensions are conformally natural with respect to appropriate Möbius transformations. That is, if A is a Möbius transformation of $\mathbb{H}^2$, or of $\mathbb{H}^3$ preserving $\mathbb{H}^2$, then $E_j(A\circ f)=A\circ E_j(f)$, $j=2,3$. Likewise, if A is a Möbius transformation of $\mathbb{R}$ then $E_1(A\circ f)=A\circ E_1(f)$. For convenience we refer to such Möbius transformations as *admissible*.

The reason why this property holds is that the curve Γ and the surface Σ, used to define the extensions, can themselves be obtained as the images of the semicircle Γ_0 and the hemisphere Σ_0 under a family of admissible Möbius transformations. Indeed, we could have constructed the extensions in this way from the outset, but the geometry would not have been as clear.

For the function f defined on $(-1, 1)$ let $M(f, x)$ be the "best Möbius approximation" to f at x. That is, $M(f, x)$ is the Möbius transformations of $\mathbb{R}$ with $M(f, x) = f(x)$, $M'(f, x) = f'(x)$ and $M''(f, x) = f''(x)$; see [T]. If A is any Möbius transformation of $\mathbb{R}$ then $M(A \circ f, x) = A \circ M(f, x)$. Such an A extends to a Möbius transformation of $\mathbb{H}^2$ and then to a Möbius transformation of $\mathbb{H}^3$ preserving $\mathbb{H}^2$ in $\mathbb{H}^3$. [So does $M(f, x)$.] We continue to denote all these extensions by the same letter.

Recall that the semicircle Γ_0 in $\mathbb{H}^2$ over $(-1, 1)$ is the envelope of the family of horocycles $H(x, \varrho_0(x))$ where the support function is

$$\varrho_0(x) = \log((1 + x^2)/(1 - x^2)).$$

Recall also that $T_0(x)$ is the point of tangency of Γ_0 and $H(x, \varrho_0(x))$. It is easy to check that if A is any Möbius transformation of $\mathbb{H}^2$ then $A(H(x, \varrho_0(x))$ has horospheric radius

$$\varrho(x) = \log(1 + (Ax)^2) - \log(1 - x^2) - \log|A'(x)|.$$

In particular, if we take $A = M(f, x)$, as a Möbius transformation of $\mathbb{H}^2$, then with $h = f^{-1}$, $u = Ax = f(x)$, we get

$$\varrho(u) = \log(1 + u^2) + \log h'(u) - \log(1 - h(u)^2).$$

This is precisely the support function defined in (4.2) for the curve Γ. That is, the map $T_0(x) \mapsto M(f, x)(T_0(x))$ realizes Γ as the image of Γ_0 under the family of best Möbius approximations to f. Extending $M(f, x)$ to be a Möbius transformation of $\mathbb{H}^3$ also realizes Σ as the corresponding image of Σ_0. Here the map is $T_0(x) \mapsto M(f, x)(T_0(z))$, $z \in \Omega_0$, where $T_0(z)$ and $T_0(x)$ are on the same latitude on Σ_0. (This is an alternative description of the mapping $\Phi: \Sigma_0 \to \Sigma$ used earlier in the proof of Theorem 1). The same thing works for the 1-parameter family of surfaces Σ_0^τ, Σ^τ.

From this it is easy to see that the extensions $E_j(f)$ are conformally natural. The extension $E_2(f)$ is defined via the reflection Λ_Σ in the surface Σ generated by the curve Γ. Let A be an admissible Möbius transformation which maps the image of f to a finite interval. Suppose that the extension $E_2(A \circ f)$ is defined via the reflection $\Lambda_{\Sigma'}$ in the surface Σ' generated by the curve Γ'. The relation

$$M(A \circ f, x) T_0(x) = A(M(f, x)) T_0(x)$$

implies that $\Gamma' = A(\Gamma)$ and hence that $\Sigma' = A(\Sigma)$. Thus to extend $A \circ f$ we reflect in $\Sigma' = A(\Sigma)$. Furthermore, because A is a hyperbolic isometry we also have $\Lambda_{\Sigma'} \circ A = A \circ \Lambda_\Sigma$. We conclude that $E_2(A \circ f) = A \circ E_2(f)$ and a fortiori that $E_1(A \circ f) = A \circ E_1(f)$. The relation $E_3(A \circ f) = A \circ E_3(f)$ follows in a similar way. With some additional argument one can show directly that the conformal naturality holds for Möbius transformations A which map the image of f to an unbounded interval, but it is even easier to check this by simply shifting the construction to the circle, the disk, and the ball, as explained in the introduction. This completes the proof of Theorem 1.

5 Remarks

First we note what happens in the extreme cases of $t=0$, $t=1$ in

$$-\frac{4t}{1-t} \leqq (1-x^2)Sf(x) \leqq \frac{4t}{1+t}.$$

When $t=0$ this reduces to $Sf(x)\equiv 0$ and this implies that f is a Möbius transformation. As $t\to 1$ the only condition is the upper bound $(1-x^2)^2 Sf(x) \leqq 2$. If strict inequality holds here our calculations, together with the results of Epstein, apply to the extent that one can conclude that f has a conformally natural *homeomorphic* extension to $\mathbb{H}^2$ and $\mathbb{H}^3$. The constant cannot be improved, as shown by the function $f(x)=\log\frac{1+x}{1-x}$ which has $(1-x^2)^2 Sf(x)=2$.

Next, notice that there is an asymmetry in the upper and lower bounds. The theorem will apply, for example, if it is known that $-\infty < -c \leqq (1-x^2)^2 Sf(x) \leqq 0$, so a negative Schwarzian appears to be a good feature if a quasiconformal extension is expected. The sign of the Schwarzian comes up in another way which we think is interesting to point out. We have no further applications right now, so we will not include the relevant calculations. Using the notation of the previous sections, we can consider the conformal metric $e^{2\psi}|d\zeta|^2$ on the domain Ω, where $\psi(\zeta)=\varrho(\zeta)-\log(1-|\zeta|^2)$. In Epstein's terminology, ψ is called the *modified support function* associated with the surface Σ. This gives a complete metric of negative curvature on Ω which is comparable to the Poincaré metric of Ω. In fact, for the Gauss curvature $\kappa(\psi)$ of $e^{2\psi}|d\zeta|^2$ one can show that

$$\frac{-4}{1-t} \leqq \kappa(\psi) \leqq \frac{-4}{1+t},$$

and that $\kappa(\psi)$ tends to -4 on $\partial\Omega\backslash\{-1,1\}$. The curvature is $\equiv -4$ if and only if the function f is a Möbius transformation. On the interval $-1<u<1$, we have

$$f^*(e^{2\psi}du^2)=(1-x^2)^2\,dx^2,$$

see (4.2). Here, the curvature can be expressed quite simply in terms of the Schwarzian of f as

$$\kappa(\psi)(u)=-4+(1-x^2)^2 Sf(x),$$

where $u=f(x)$. Thus a negative Schwarzian means that f decreases curvature along $(-1,1)$. One can also get a fairly simple expression for $\kappa(\psi)$ at any point on Ω in terms of the Schwarzian along $(-1,1)$, but we will not give it here.

In principle there are many more functions satisfying the hypothesis of Theorem 1 then there are analytic functions in the disk satisfying the Ahlfors-Weill condition. For example the function f on $(-1,1)$ with

$$f'(x)=\left(\frac{x^2+a}{1-x}\right)^{\alpha}$$

satisfies the hypothesis of Theorem 1 if $a\in(0,1)$ and $\alpha\in(0,1)$ is sufficiently small, but f has no analytic extension to the disk.

Finally, the reader who is familiar with Epstein's work and with injectivity criteria for functions analytic in the disk may wonder if there is an analogue of

Theorem 1 with f''/f' replacing the Schwarzian. So do we. One can bring f''/f' alone into the picture by changing the support function ϱ in (4.2) to

$$\varrho(u)=\log(1-u^2)-\log(1-h(u)^2),$$

in earlier notation. This choice of ϱ results from using h to pull back the metric $f'dx/(1-x^2)$ rather than the metric $dx/(1-x^2)$ as was done before. One can proceed as before, to a point, but trouble comes in showing that the extension $\tilde{f}$ of f to the unit disk is quasiconformal. This doesn't arise in Epstein's work since the function is defined on the disk to begin with. Whether this is a defect in the way we construct the extension, or that something more subtle is going on we do not know.

References

[AW] Ahlfors, L.V., Weill, G.: A uniqueness theorem for Beltrami equations. Proc. Am. Math. Soc. **13**, 975–978 (1962)

[C1] Chuaqui, M.: A unified approach to injectivity criteria in the disk and in simply connected domains (to appear)

[C2] Chuaqui, M.: The Schwarzian derivative and quasiconformal reflections in S^n (to appear)

[CO] Chuaqui, M., Osgood, B.: Sharp distortion theorems associated with the Schwarzian derivative (to appear)

[DE] Douady, A., Earle, C.: Conformally natural extension of homeomorphisms of the circle. Acta Math. **157**, 23–47 (1986)

[E1] Epstein, C.: The hyperbolic Gauss map and quasiconformal reflection. J. Reine Angew. Math. **372**, 96–135 (1986)

[E2] Epstein, C.: Envelopes of homospheres and Weingarten surfaces in hyperbolic 3-space (to appear)

[L] Lehto, O.: Univalent functions and Teichmüller spaces. Berlin Heidelberg New York: Springer 1987

[N] Nehari, Z.: The Schwarzian derivative and schlicht functions. Bull. Am. Math. Soc. **55**, 545–551 (1949)

[OS1] Osgood, B., Stowe, D.: The Schwarzian derivative and conformal mapping of Riemannian manifolds (to appear)

[OS2] Osgood, B., Stowe, D.: A generalization of Nehari's univalence criterion. Comment. Math. Helv. **65**, 234–242 (1990)

[T] Thurston, W.: Zippers and univalent functions. In: Baernstein II, A., Drasin, B., Duren, P., Mardin, A. (eds.): The Bieberbach Conjecture. (Math. Surv. Monogr., vol. 21, pp. 185–197) Providence, RI: Am. Math. Soc. 1986